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Braided vector fields on a quantum hyperboloid via the quantum group $U_q(sl(2))$

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Abstract
We introduce a new method of defining a space of 'braided vector fields' on a quantum hyperboloid. Our method is based on the Lyubashenko–Sudbery construction (see Lyubashenko and Sudbery 1998 *J. Math. Phys.* **39** 3487 and references therein) and it consists in realizing its generators (braided analogs of hyperbolic infinitesimal rotations) via those of the quantum group $U_q(sl(2))$. Our main result consists in showing that the space of braided vector fields is a projective module over the coordinate algebra of the quantum hyperboloid.

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1. Introduction

Let us consider a sphere $S_r^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = r^2, r > 0\}$ embedded in the Euclidean space $\mathbb{R}^3 \cong so(3)^*$ as an orbit of action of the group $SO(3)$. Also, we endow the coordinate algebra $\mathbb{K}[\mathbb{R}^3]$ of the space $\mathbb{R}^3$ with a $SO(3)$-covariant Poisson bracket:

$$\{x, y\} = z, \qquad \{y, z\} = x, \qquad \{z, x\} = y.$$

Hereafter, $\mathbb{K} = \mathbb{C}$ (or $\mathbb{R}$) is the ground field, the notation $\mathbb{K}[\mathcal{M}]$ stands for the coordinate algebra of a given regular affine algebraic variety $\mathcal{M}$.

Then the operators $X = \{x, \cdot\}$, $Y = \{y, \cdot\}$, $Z = \{z, \cdot\}$ are infinitesimal rotations. Their explicit form is

$$X = z\partial_y - y\partial_z, \qquad Y = x\partial_z - z\partial_x, \qquad Z = y\partial_x - x\partial_y.$$

They are tangent to the spheres S_r^2 and subject to the relation

$$xX + yY + zZ = 0. \tag{1.1}$$

Consider the coordinate algebra of the sphere S_r^2

$$\mathbb{K}\left[S_r^2\right] = \mathbb{K}[\mathbb{R}^3]/\langle x^2 + y^2 + z^2 - r^2\rangle.$$

Hereafter $\langle I \rangle$ stands for the two-sided ideal generated by a set *I*. The space $\mathrm{Vect}(S_r^2)$ of all vector fields on a sphere (with coefficients from $\mathbb{K}[S_r^2]$), treated as a $\mathbb{K}[S_r^2]$-module, is the quotient

$$M = K[S_r^2]^{\oplus 3}/\overline{M}$$

of the free $\mathbb{K}[S_r^2]$-module $\mathbb{K}[S_r^2]^{\oplus 3}$ over the submodule $\overline{M} = \{\varphi(xX+yY+zZ), \forall \varphi \in \mathbb{K}[S_r^2]\}$.

It is not difficult to see that the module $\overline{M}$ is projective. Indeed the matrix

$$\bar{e} = \frac{1}{r^2}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\begin{pmatrix} x & y & z \end{pmatrix}$$

defines an idempotent such that $\overline{M} = \mathbb{K}[S_r^2]^{\oplus 3}\bar{e} \subset \mathbb{K}[S_r^2]^{\oplus 3}$. (Hereafter all modules are left.) Therefore the $\mathbb{K}[S_r^2]$-module M can be realized as a submodule:

$$M = \mathbb{K}[S_r^2]^{\oplus 3} e \subset \mathbb{K}[S_r^2]^{\oplus 3},$$

where $e = Id - \bar{e}$ is the complementary idempotent. We call the $\mathbb{K}[S_r^2]$-module M *tangent*. In contrast to other $\mathbb{K}[S_r^2]$-modules on the tangent module, an action $M \otimes \mathbb{K}[S_r^2] \longrightarrow \mathbb{K}[S_r^2]$ is defined. This action consists in applying a vector field to a function.

In a similar manner the tangent module on a hyperboloid

$$\mathrm{H}_\rho^2 = \{(b, g, c) \in \mathbb{R}^3 | g^2 + 4bc = \rho^2, \rho \neq 0\} \tag{1.2}$$

can be defined. The main goal of this paper is to explicitly define q-analogs of vector fields tangent to this hyperboloid. Note that if $\mathbb{K} = \mathbb{R}$ and ρ is real, we get a one-sheeted hyperboloid. If ρ is purely imaginary, we get a two-sheeted hyperboloid. However, if $\mathbb{K} = \mathbb{C}$ we allow ρ to take any non-trivial value. Let us emphasize that we prefer to deal with a q-analog of a hyperboloid since a q-analog of a sphere (the so-called Podles sphere) cannot be realized as a real algebra.

Our hyperboloid is treated to be an orbit $\mathrm{H}_\rho^2 \hookrightarrow sl(2)^*$ of the coadjoint action of the Lie algebra $sl(2)$. Vector fields arising from this coadjoint action are tangent to all orbits in $sl(2)^*$ and they will be called *tangent vector fields*. (Also, they are the Poisson vector field w.r.t. the linear Poisson–Lie bracket defined on the space $sl(2)^*$.) Below, we introduce braided analogs of these vector fields. They are in a sense tangent to a quantum hyperboloid and form a projective module over the algebra $\mathbb{K}_q[\mathrm{H}_\rho^2]$.

To define the space $\mathrm{Vect}(\mathrm{H}_\rho^2)$ of all vector fields on a hyperboloid, we apply the above scheme to the space $\mathbb{R}^3 \cong sl(2)^*$ endowed with an action of the group $SL(2)$. The corresponding hyperbolic infinitesimal rotations B, G, C tangent to the hyperboloid (1.2) span the space $\mathrm{Vect}(\mathrm{H}_\rho^2)$ and are subject to an analog of the relation (1.1):

$$2cB + gG + 2bC = 0. \tag{1.3}$$

A quantum analog of the algebra $\mathbb{K}[\mathrm{H}_\rho^2]$ is well known (it can easily be obtained from the property that this algebra is $U_q(sl(2))$-covariant). Moreover, this algebra can be introduced to be an appropriate quotient of the so-called q-Minkowski space algebra $\mathbb{K}_q[\mathbb{R}^4]$ (as defined in [CW1, CW2, OS, MM, M, K]) which is a particular case of reflection equation algebra (REA). These vector fields are associated with the algebra $\mathbb{K}_q[\mathrm{H}_\rho^2]$.

Let $\mathcal{L}$ be the space spanned by the generators of $\mathbb{K}_q[\mathbb{R}^4]$ and $\mathcal{SL} = \mathrm{span}(b, g, c)$ be a three-dimensional subspace of the space $\mathcal{L}$. We need a braided analog $[\,,\,]_q : \mathcal{SL}^{\otimes 2} \longrightarrow \mathcal{SL}$ of the $sl(2)$ Lie bracket. By using this q-bracket we define 'braided tangent vector fields'

B_q, G_q, C_q following the classical pattern. They are tangent to the quantum hyperboloid and subject to an analog of the relation (1.3):

$$(1+q^{-2})bC_q + gG_q + (q^2+1)cB_q = 0. \tag{1.4}$$

The main difficulty is then of extending these braided vector fields well defined on $\mathcal{SL}$ to $\mathbb{K}_q[\mathbb{R}^3]$ preserving the relation (1.4).

In [A] and [DG] a non-constructive method is suggested using a family of $U_q(sl(2))$-covariant projectors P_k ($k \in \mathbb{N}^*$ i.e.: k is a strictly positive integer) defined as polynomials in elements using a braiding of the Birman–Murakamin–Wenzl type. (A way of constructing such operators P_k is described in [OP].) As a result, our braided vector fields are well defined on $\mathbb{K}_q[\mathbb{R}^3]$ but it is very difficult to calculate their action on elements of $\mathbb{K}_q[\mathbb{R}^3]$ of higher degree than 1.

Thus, in this paper we propose, using the Lyubashenko–Sudbery embedding of $sl(2)_q$ in the QG $U_q(sl(2))$ (see [W] and [LS]), to define this extension and then to deduce an explicit representation of our vector fields knowing the representation theory of $sl(2)_q$ in [DS].

Note that these braided vector fields are useful tools to define q-analogs of the Laplace and Maxwell operators on non-commutative algebras (see [DG]), and the main interest of our construction is that it helps to better understand the action of our braided vector fields on $\mathbb{K}_q[\mathbb{R}^3]$. Consequently, we think it could facilitate the study of the q-analogs of the Laplace and Maxwell operators defined in [DG].

2. Vector fields on a classical hyperboloid

Let us consider the coordinate algebra $\mathbb{K}[\mathbb{R}^3]$ endowed with a basis $\{b, g, c\}$ and a $SL(2)$-covariant Poisson bracket:

$$\{g, b\} = 2b, \qquad \{g, c\} = -2c, \qquad \{b, c\} = g.$$

The corresponding Poisson fields are

$$B = \{b, .\} = g\partial_c - 2b\partial_g, \qquad G = \{g, .\} = 2b\partial_b - 2c\partial_c, \qquad C = \{c, .\} = -g\partial_b + 2c\partial_g.$$

They are tangent to hyperboloids H^2_ρ and subject to the relation

$$2cB + gG + 2bC = 0. \tag{2.1}$$

The commutation relations for B, G, C are

$$\begin{cases} GB - BG = 2B \\ BC - CB = G \\ CG - GC = 2C. \end{cases} \tag{2.2}$$

These operators in the basis $\{b, -g, -c\}$ are

$$B = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad G = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}. \tag{2.3}$$

Consider the coordinate algebra of a hyperboloid H^2_ρ:

$$\mathbb{K}\left[\mathrm{H}^2_\rho\right] = \mathbb{K}[\mathbb{R}^3]/\langle 4bc + g^2 - \rho^2\rangle, \qquad \rho \neq 0.$$

The space Vect $\left(\mathrm{H}^2_\rho\right)$ of all vector fields on this hyperboloid (with a coefficient from $\mathbb{K}[\mathrm{H}^2_\rho]$), treated as a $\mathbb{K}\left[\mathrm{H}^2_\rho\right]$-module, is the quotient

$$M = \mathbb{K}\left[\mathrm{H}^2_\rho\right]^{\oplus 3} / \overline{M}$$

of the free $\mathbb{K}[\mathrm{H}^2_\rho]$-module $\mathbb{K}[\mathrm{H}^2_\rho]^{\oplus 3}$ over the submodule

$$\overline{M} = \{\varphi(2cB + gG + 2bC), \forall\varphi \in \mathbb{K}[\mathrm{H}^2_\rho]\}.$$

It is not difficult to see that the module $\overline{M}$ is projective. Indeed, the matrix

$$\bar{e} = \frac{1}{\rho^2}\begin{pmatrix}2c\\ g\\ 2b\end{pmatrix}\begin{pmatrix}b & g & c\end{pmatrix}$$

defines an idempotent such that $\overline{M} = \mathbb{K}[\mathrm{H}^2_\rho]^{\oplus 3}\bar{e}$. Therefore the $\mathbb{K}[\mathrm{H}^2_\rho]$-module M can be realized as a submodule

$$M = \mathbb{K}[\mathrm{H}^2_\rho]^{\oplus 3} e \subset \mathbb{K}[\mathrm{H}^2_\rho]^{\oplus 3},$$

where $e = Id - \bar{e}$ is the complementary idempotent.

The $\mathbb{K}[\mathrm{H}^2_\rho]$-module M is called *tangent*. In contrast to other $\mathbb{K}[\mathrm{H}^2_\rho]$-modules on the tangent module, an action $M \otimes \mathbb{K}[\mathrm{H}^2_\rho] \to \mathbb{K}[\mathrm{H}^2_\rho]$ is defined. It consists in applying a vector field to a function.

Let us fix an integer $k \in \mathbb{N}^*$ and consider the vector space:

$$V_k = \mathrm{span}(b^k, C(b^k), C^2(b^k), \ldots, C^{2k}(b^k)).$$

(Observe that $\dim(V_k) = 2k + 1$.)

We denote by $\mathrm{diag}(a_1, a_2, \ldots, a_n)$ the $n \times n$ diagonal matrices, by $\mathrm{diag}_+(a_1, a_2, \ldots, a_{n-1})$ the $n \times n$ matrices having possibly non-zero elements $a_1, \ldots, a_{n-1}$ only on the first overdiagonal and by $\mathrm{diag}_-(a_1, a_2, \ldots, a_{n-1})$ the $n \times n$ matrices having possibly non-zero elements $a_1, \ldots, a_{n-1}$ only on the first subdiagonal.

In the basis $\left\{b^k, \frac{1}{1!}C(b^k), \frac{1}{2!}C^2(b^k), \ldots, \frac{1}{(2k)!}C^{2k}(b^k)\right\}$ of V_k the operators B, G, C are, respectively, represented by the $(2k + 1) \times (2k + 1)$ matrices B^k, G^k, C^k:

$$B^k = \mathrm{diag}_+(2k, 2k - 1, 2k - 2, \ldots, 1) \tag{2.4}$$

$$G^k = \mathrm{diag}(2k, 2k - 2, 2k - 4, \ldots, -2k) \tag{2.5}$$

$$C^k = \mathrm{diag}_-(1, 2, 3, \ldots, 2k). \tag{2.6}$$

Thus, the vector fields B, G, C being restricted to the component V_k realize the finite-dimensional representation $\rho_k : sl(2) \longrightarrow End(V_k)$ of the algebra $sl(2)$ defined by the matrices (2.4)–(2.6). The restriction of (2.2) on V_k gives us

$$\begin{cases} G^k B^k - B^k G^k = 2B^k \\ B^k C^k - C^k B^k = G^k \\ C^k G^k - G^k C^k = 2C^k. \end{cases} \tag{2.7}$$

So the operator B can be defined as a collection $\{B^k,\ k \in \mathbb{N}^*\}$ and similarly for those G and C. Besides, it follows from the definition that $B(1) = G(1) = C(1) = 0$. Also, as we said the operators B, G, C are subject to the relation (2.1) and they generate a projective module M. Our main goal consists in constructing q-analogs of these operators.

3. Quantum hyperboloid via REA

There exists different ways of introducing quantum orbits. In this section, we define a quantum hyperboloid algebra as a quotient of the so-called REA. First, let us introduce this algebra.

Let V be a vector space over the ground field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ and $\mathrm{R} : V^{\otimes 2} \to V^{\otimes 2}$ be an invertible operator satisfying the braid equation

$$(\mathrm{R} \otimes \mathrm{Id})(\mathrm{Id} \otimes \mathrm{R})(\mathrm{R} \otimes \mathrm{Id}) = (\mathrm{Id} \otimes \mathrm{R})(\mathrm{R} \otimes \mathrm{Id})(\mathrm{Id} \otimes \mathrm{R}).$$

Such a solution is called *braiding*. (Note that R is usually denoted by $\hat{R}$.) If moreover, R satisfies the relation

$$(q\,Id - \mathrm{R})(q^{-1}Id + \mathrm{R}) = 0, \quad q \in \mathbb{K}$$

R is called a *Hecke symmetry* for $q \neq 1$ and an *involutive symmetry* for $q = 1$. In the following q is assumed to be generic. This means that q can take any value from $\mathbb{K}$ apart from those from a countable subset (which does not contain 1). Observe that for the QG $U_q(sl(2))$ (we will recall the definition of $U_q(sl(2))$ at the beginning of section 4) generic means $q \notin \{0, i, -i\}$ (see [RT]).

Fix a basis $\{x_i\} \in V,\ 1 \leqslant i \leqslant n = \dim V$ and the corresponding basis $\{x_i \otimes x_j\}$ in the space $V^{\otimes 2}$. Then the operator R can be identified with a matrix.

Example 1. Let $n = 2$. Fix a basis in which the standard generators of the QG $U_q(sl(2))$ have the classical form

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then the Hecke symmetry, which is the image of the universal quantum R-matrix composed with the usual flip, is the following:

$$R_q = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{3.1}$$

Example 2. A natural generalization of the previous example for $n > 2$ is

$$\mathrm{R} = \sum_{i,j} q^{\delta_{ij}}\, h_i^j \otimes h_j^i + \sum_{i<j} (q - q^{-1})\, h_i^i \otimes h_j^j, \tag{3.2}$$

where $\{h_i^j\}$ is the base in the space of left endomorphisms of the basic space V such that $h_i^j(x_k) = \delta_k^j\, x_i$. In the matrix form, it reads

$$\mathrm{R}_{ij}^{kl} = q^{\delta_{kl}} \delta_i^l \delta_j^k + (q - q^{-1})\Theta(j - i)\delta_i^k \delta_j^l,$$

where $\Theta(i) = 1$ for $i > 0$ and $\Theta(i) = 0$ for $i \leqslant 0$.

Definition 3. *The algebra generated by the unity and generators l_i^j, $1 \leqslant i, j \leqslant n$ subject to the system*

$$\mathrm{R}(L \otimes \mathrm{Id})\mathrm{R}(\mathrm{L} \otimes \mathrm{Id}) - (\mathrm{L} \otimes \mathrm{Id})\mathrm{R}(\mathrm{L} \otimes \mathrm{Id}))\mathrm{R} = 0, \tag{3.3}$$

where $L = \|l_i^j\|$ is a matrix with the entries l_i^j is called the reflection equation algebra (REA). It is denoted by $\mathcal{L}(q)$.

Example 4. Go back to example 1. So, R is the form of the matrix (3.1).

Let $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} l_1^1 & l_1^2 \\ l_2^1 & l_2^2 \end{pmatrix}$ be the matrix coming in (3.3):

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$
$$- \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$
$$\times \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} = 0.$$

By explicitly computing this system we obtain

$$\begin{aligned} &qab - q^{-1}ba = 0, \qquad q(bc-cb) = (q-q^{-1})a(d-a), \\ &qca - q^{-1}ac = 0, \qquad q(cd-dc) = (q-q^{-1})ca, \\ &ad - da = 0, \qquad q(db-bd) = (q-q^{-1})ab. \end{aligned} \tag{3.4}$$

The algebra $\mathcal{L}(q)$ corresponding to this Hecke symmetry R_q is called the *q-Minkowski space algebra*. In this case it is also denoted by $\mathbb{K}_q[\mathbb{R}^4]$.

There exists an automorphism $C : V \to V$ (in a matrix form $C = (C_i^j)$) such that the elements $\mathrm{Tr}\,(CL^k)$, $k = 0, 1, 2, \ldots$ are central in the algebra $\mathcal{L}(q)$ (here Tr is the usual trace). The quantities $\mathrm{Tr}\,(CL^k)$ are also denoted by $\mathrm{Tr}_q\, L^k$ and called q-trace (or quantum trace) of the matrices L^k. We consider the q-trace Tr_q as an operator from $\mathrm{Mat}_n(A)$ to A where $A = \mathcal{L}(q)$. Note that the matrix C is unique up to a factor.

By assuming R to be as in example 1 and with an appropriate normalization of this matrix we obtain $C = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}$ and therefore $\mathrm{Tr}_q\; L = q^{-1}\,a + q\,d$.

Now, we rewrite the system (3.4) in the basis $\{\ell, b, g, c\}$ where $\ell = q^{-1}a + qd$, $g = a - d$:

$$\begin{aligned} &q^2gb - bg = -(q-q^{-1})\ell b, \qquad b\ell = \ell b, \\ &(q^2+1)(bc-cb) + (q^2-1)g^2 = -(q-q^{-1})\ell g, \qquad g\ell = \ell g, \\ &q^2cg - gc = -(q-q^{-1})\ell c, \qquad c\ell = \ell c. \end{aligned} \tag{3.5}$$

Hereafter, $n_q = \frac{q^n - q^{-n}}{q - q^{-1}}$ is the q-analog of the number $n \in \mathbb{N}$.

Now, we introduce the algebra $\mathbb{K}_q[\mathbb{R}^3] = \mathbb{K}_q[\mathbb{R}^4]/\langle\ell\rangle$ which is a braided analog of the coordinate algebra $\mathbb{K}[\mathbb{R}^3]$. It is generated by three elements b, g, c subject to

$$q^2gb - bg = 0, \qquad (q^2+1)(bc-cb) + (q^2-1)g^2 = 0, \qquad q^2cg - gc = 0. \tag{3.6}$$

The generating spaces span(a, b, c, d) and span(b, g, c) of the algebras $\mathbb{K}_q[\mathbb{R}^4]$ and $\mathbb{K}_q[\mathbb{R}^3]$ are, respectively, denoted by $\mathcal{L}$ and $\mathcal{SL}$.

Consider the central element $\mathrm{Tr}_q L^2 \in \mathbb{K}_q[\mathbb{R}^4]$. In the basis $\{b, g, c, \ell\}$ its explicit form is

$$\mathrm{Tr}_q L^2 = q^{-1}bc + qcb + 2_q^{-1}(g^2 + \ell^2).$$

Its image in the algebra $\mathbb{K}_q[\mathbb{R}^3]$ is $q^{-1}bc + 2_q^{-1}g^2 + qcb$. In what follows its multiple

$$q^{-1}2_qbc + g^2 + q2_qcb$$

is denoted by Cas_q. It is a central element in this algebra. So, it is a braided analog of the $sl(2)$ Casimir element $\mathrm{Cas} = 2bc + g^2 + 2cb$. The quotient of the algebra $\mathbb{K}_q[\mathbb{R}^3]$ over the ideal generated by the element $\mathrm{Cas}_q - \rho^2$ is called the *quantum hyperboloid* if $\rho \neq 0$. It is $\mathbb{K}_q\left[\mathrm{H}^2_\rho\right]$. Observe that Cas_q is the unique quadratic element (up to a factor) in $\mathbb{K}_q[\mathbb{R}^3]$ which is $U_q(sl(2))$-invariant.

4. The embedding of $sl(2)_q$ in the QG $U_q(sl(2))$

Recall that the QG $U_q(sl(2))$ is generated by the unit and four generators K, K^{-1}, X, Y subject to the relations

$$\begin{gathered} K^\epsilon K^{-\epsilon} = 1, \qquad K^\epsilon X = q^{2\epsilon} X K^\epsilon, \qquad K^\epsilon Y = q^{-2\epsilon} Y K^\epsilon, \\ XY - YX = \frac{K - K^{-1}}{q - q^{-1}}, \qquad \epsilon = \pm 1. \end{gathered} \tag{4.1}$$

There exists a coproduct Δ and an antipode S which (together with the standard counit) endow this algebra with a Hopf structure:

$$\Delta(K^\epsilon) = K^\epsilon \otimes K^\epsilon, \qquad \Delta(X) = X \otimes K^{-1} + 1 \otimes X, \qquad \Delta(Y) = Y \otimes 1 + K \otimes Y \tag{4.2}$$

$$S(K^\epsilon) = K^{-\epsilon}, \qquad S(X) = -XK, \qquad S(Y) = -K^{-1}Y. \tag{4.3}$$

Let us define an action of the QG $U_q(sl(2))$ on the space $\mathcal{SL}$ as follows:

$$X(b) = 0, \qquad X(g) = -2_q b, \qquad X(c) = qg, \tag{4.4}$$

$$K^\epsilon(b) = q^{2\epsilon} b, \qquad K^\epsilon(g) = g, \qquad K^\epsilon(c) = q^{-2\epsilon} c, \tag{4.5}$$

$$Y(b) = -g, \qquad Y(g) = q^{-1} 2_q c, \qquad Y(c) = 0. \tag{4.6}$$

The reader can easily check that the structure of the algebra $\mathbb{K}_q[\mathbb{R}^3]$ is compatible with the action of the QG $U_q(sl(2))$ extended to higher power of $\mathcal{SL}$ via the coproduct Δ. In order to do this, it suffices to check that the system (3.6) is invariant w.r.t. the QG $U_q(sl(2))$.

Considering the Hopf algebra $(U_q(sl(2)), \Delta, S)$, let us conceive the 'quantum adjoint' action of the QG on itself defined as follows (hereafter, we use the Sweedler notation):

$$\forall\, U, V \in U_q(sl(2)) : \quad U(V) = U_1 V S(U_2) \qquad \text{with} \quad \Delta(U) = U_1 \otimes U_2.$$

Proposition 5. *The vector space $sl(2)_q$ generated in the algebra $U_q(sl(2))$ by X_+, X_0, X_- with*

$$X_+ = X, \qquad X_0 = q^2 XY - YX, \qquad X_- = qK^{-1}Y$$

is closed under the action of the QG $U_q(sl(2))$ and is by consequence isomorphic to the matrix representation.

Proof. See [LS]. However, for the convenience of the reader, we can prove this proposition with our notations. The formulae that we use for this proof are (4.1)–(4.3).

We place ourselves in $U_q(sl(2))$ with the coproduct Δ and its associated antipode S. We can then define an ad-action on $U_q(sl(2))$ using Δ and S.

Let $X_+ = X$:

$$X(X_+) = X(-XK) + XXK = -X^2K + X^2K = 0.$$

Let $X_0 = -Y(X_+)$:

$$Y(X_+) = KX(-K^{-1}Y) + YX = -(q^2XY - YX).$$

So $X_0 = q^2XY - YX$.

Let now $X_- = \frac{q}{2_q}Y(X_0)$:

$$Y(X_0) = q^2(YX - XY)Y + Y(XY - YX).$$

So

$$(q - q^{-1})Y(X_0) = q^2(K^{-1} - K)Y + Y(K - K^{-1}) = K^{-1}(q^2 - q^{-2})Y,$$

and then $X_- = qK^{-1}Y$.

It is easy to see that

$$\begin{aligned}
&K(X_+) = q^2X_+, \qquad K(X_0) = X_0, \qquad K(X_-) = q^{-2}X_-\\
&Y(X_-) = q\left(Y\left(-K^{-1}Y\right) + YK^{-1}Y\right) = 0\\
&X(X_-) = q\left(K^{-1}Y(-XK) + XK^{-1}YK\right) = q(q^2XY - YX) = qX_0\\
&X(X_0) = q^2XY(-XK) + q^2X^2YK - YX(-XK) - XYXK\\
&X(X_0) = K^{-1}\frac{q^2XK - q^2XK^{-1} - KX + K^{-1}X}{q - q^{-1}}K = -2_qX_+ .
\end{aligned}$$

□

Proposition 6. *The generators X_+, X_0, X_- satisfy*

$$\begin{cases}
q^2X_0X_+ - X_+X_0 = CmrX_+\\
(q^2+1)(X_+X_- - X_-X_+) + (q^2-1)X_0^2 = CmrX_0\\
q^2X_-X_0 - X_0X_- = CmrX_-
\end{cases}$$

with Cmr being a central element in the QG $U_q(sl(2))$:

$$Cmr = (q^4 - q^2 + 1)XY - q^2YX - \frac{1}{q - q^{-1}}K + \frac{q^4}{q - q^{-1}}K^{-1}.$$

Proof. See [LS]. □

Corollary 7. *Let us consider an irreducible matrix representation π of the QG $U_q(sl(2))$ then*

$$\begin{cases}
q^2\pi(X_0)\pi(X_+) - \pi(X_+)\pi(X_0) = Cmr_\pi\pi(X_+)\\
(q^2+1)\left(\pi(X_+)\pi(X_-) - \pi(X_-)\pi(X_+)\right) + (q^2-1)\pi(X_0)^2 = Cmr_\pi\pi(X_0)\\
q^2\pi(X_-)\pi(X_0) - \pi(X_0)\pi(X_-) = Cmr_\pi\pi(X_-)
\end{cases} \tag{4.7}$$

with $Cmr_\pi \in \mathbb{K}$.

5. Braided analog of tangent vector fields

Let us endow the space $\mathcal{SL}$ with a braided analog of the Lie bracket $sl(2)$. In order to do so, we extend the action of the QG $U_q(sl(2))$ to the space $\mathcal{SL}\otimes\mathcal{SL}$ and decompose it into a direct sum of irreducible $U_q(sl(2))$ submodules $\mathcal{SL}\otimes\mathcal{SL} = V_0 \oplus V_1 \oplus V_2$, where the subscript stands for the spin. Then the operator

$$[\ ,\ ] : \mathcal{SL}\otimes\mathcal{SL} \to \mathcal{SL}$$

is a $U_q(sl(2))$-morphism iff it is trivial on the components V_0 and V_2, and it is an isomorphism between V_1 and $\mathcal{SL}$. By this condition, the bracket is defined in the unique (up to a factor) way.

Its multiplication table is as follows:

$$[b,b]=0,\qquad [b,g]=-\omega b,\qquad [b,c]=\omega\frac{q}{2_q}g,\qquad [g,b]=\omega q^2 b,$$
$$[g,g]=\omega(q^2-1)g,\qquad [g,c]=-\omega c,\qquad [c,b]=-\omega\frac{q}{2_q}g,$$
$$[c,g]=\omega q^2 c,\qquad [c,c]=0.$$

Here ω is an arbitrary factor.

Introduce q-analogs of the adjoint operators as follows $B_q=ad(b),\ G_q=ad(g), C_q=ad(c)$, where the action ad is defined via the above bracket. These operators in the basis $\{b,-g,-q^{-1}c\}$ are

$$B_q=\omega\begin{pmatrix}0&1&0\\0&0&2_q^{-1}\\0&0&0\end{pmatrix}\qquad G_q=\omega\begin{pmatrix}q^2&0&0\\0&q^2-1&0\\0&0&-1\end{pmatrix}\qquad C_q=\omega\begin{pmatrix}0&0&0\\q2_q^{-1}&0&0\\0&q^3&0\end{pmatrix}. \tag{5.1}$$

Thus, these operators are subject to

$$\begin{cases}q^2G_qB_q-B_qG_q=\theta B_q\\(q^2+1)(B_qC_q-C_qB_q)+(q^2-1)G_q^2=\theta G_q\\q^2C_qG_q-G_qC_q=\theta C_q\end{cases} \tag{5.2}$$

provided $\omega=\theta(q^4-q^2+1)^{-1}$.

We deduce the classical limit (2.3) from (5.1) for $q=1,\theta=2$.

Proposition 8. *The operators B_q, G_q, C_q (defined on $\mathcal{SL}$) are subject to*

$$q^{-1}2_qbC_q+gG_q+q2_qcB_q=0. \tag{5.3}$$

Proof. It is straightforward. □

We now consider the main goal of this paper. We want to extend the operators B_q, G_q, C_q to the algebra $\mathbb{K}_q[\mathbb{R}^3]$, preserving the relation (5.3). A method is suggested in [A] and [DG]. Our method is different because we use the Lyubashenko–Sudbery construction. We get then an explicit extension formula of our operators B_q, G_q, C_q via the coproduct Δ of the QG $U_q(sl(2))$.

We can see that on $\mathcal{SL}$:

$$B_q=\tau X=\tau X_+,\qquad G_q=\tau(q^2XY-YX)=\tau X_0,$$
$$C_q=\tau qK^{-1}Y=\tau X_-\qquad\text{with}\quad \tau=2_q^{-1}\omega.$$

And $X_+,\ X_0,\ X_-$ are well defined on $\mathbb{K}_q[\mathbb{R}^3]$ because X, K, Y are defined on $\mathbb{K}_q[\mathbb{R}^3]$ via the coproduct Δ.

This leads us to the main result of this paper.

Theorem 9. *On $\mathbb{K}_q[\mathbb{R}^3]$, the operators $X_+,\ X_0,\ X_-$ are subject to*

$$q^{-1}2_qbX_-+gX_0+q2_qcX_+=0. \tag{5.4}$$

Proof. It is straightforward from (4.2), (4.4) and (4.5) that

$$\forall\, k\in\mathbb{N}^*:\quad X(b^k)=0\qquad\text{and}\qquad K(b^k)=q^{2k}b^k.$$

So b^k is a highest weight vector.

We note that $\mathrm{Cas}_q = q^{-1}2_q bX_- + gX_0 + q2_q cX_+$.

Let us show that

$$\forall k \in \mathbb{N}^* : \quad \mathrm{Cas}_q(b^k) = 0.$$

From (4.1), (4.5) and (3.6):

$$\begin{aligned}
\mathrm{Cas}_q(b^k) &= q^{-1}2_q bqK^{-1}Y(b^k) + g(q^2XY - YX)(b^k) + q2_q cX(b^k) \\
&= 2_q bK^{-1}Y(b^k) + q^2 gXY(b^k) \\
&= 2_q bq^2 YK^{-1}(b^k) + q^2 g\left(YX + \frac{K - K^{-1}}{q - q^{-1}}\right)(b^k) \\
&= q^{-2k+2}2_q bY(b^k) + q^2(2k)_q gb^k \\
&= q^{-2k+2}b(2_q Y(b^k) + (2k)_q b^{k-1}g).
\end{aligned}$$

We show then by induction on $k \in \mathbb{N}^*$ that

$$\forall\, k \in \mathbb{N}^* : \quad Y(b^k) = -2_q^{-1}(2k)_q b^{k-1}g.$$

The statement holds for $k = 1$ from (4.6).

If the statement holds for some $k \in \mathbb{N}^*$ then from (4.2), (4.5), (4.6) and (3.6):

$$\begin{aligned}
Y(b^{k+1}) = Y(b.b^k) &= Y(b)b^k + K(b)Y(b^k) \\
&= -gb^k + q^2 b\left(-2_q^{-1}\,(2k)_q\, b^{k-1}g\right) \\
&= -q^{-2k}b^k g - q^2 2_q^{-1}(2k)_q b^k g \\
&= -2_q^{-1}\left(q^{-2k}2_q + (2k)_q\, q^2\right) b^k g \\
&= -2_q^{-1}(2(k+1))_q b^k g.
\end{aligned}$$

So, the statement holds for $k + 1$ and we deduce that

$$\forall\, k \in \mathbb{N}^* : \quad \mathrm{Cas}_q(b^k) = 0.$$

Let $k \in \mathbb{N}^*$ and V_k be the $U_q(sl(2))$ submodule of $\mathcal{SL}^{\otimes k}$ with the highest vector b^k, i.e.,

$$V_k = \mathrm{span}(b^k, Y(b^k), Y^2(b^k), \ldots, Y^{2k}(b^k)).$$

We consider the following lemma.

Lemma 10.

$$\forall\, v \in \mathbb{K}_q[\mathbb{R}^3] : \qquad Y\left(\mathrm{Cas}_q\,(v)\right) = \mathrm{Cas}_q(Y(v)).$$

We conclude from this lemma that

$$\forall\, k \in \mathbb{N}^* : \quad \mathrm{Cas}_q(V_k) = 0.$$

Now, observe that for a generic q we have $\mathbb{K}_q[\mathbb{R}^3] \cong (\oplus V_k) \otimes Z$, where Z is $U_q(sl(2))$-invariant. It concludes the proof:

$$\mathrm{Cas}_q(\mathbb{K}_q[\mathbb{R}^3]) = 0.$$

It is then easy to extend B_q, G_q, C_q from $\mathcal{SL}$ to $\mathbb{K}_q[\mathbb{R}^3]$ preserving the relation (5.3). Thus, the space of all braided tangent vector fields is treated as a left $\mathbb{K}_q[\mathbb{R}^3]$-module which is the quotient of the free module $\mathbb{K}_q[\mathbb{R}^3]^{\oplus 3}$ over the submodule generated by the lhs of (5.3).

Recall $\mathrm{Cas}_q = q^{-1}2_q bc + g^2 + q2_q cb$ and observe that

$$X_+(\mathrm{Cas}_q - \rho^2) = X_0(\mathrm{Cas}_q - \rho^2) = X_-(\mathrm{Cas}_q - \rho^2) = 0 \qquad \text{for} \quad \rho \neq 0.$$

Thus, the operators X_+, X_0, X_-, and consequently our braided tangent vector fields B_q, G_q, C_q, are also defined on $\mathbb{K}_q\left[\mathrm{H}^2_\rho\right]$.

Proof of lemma 10.

$$\mathrm{Cas}_q(v) = q^{-1}2_q bqK^{-1}Y(v) + g(q^2XY - YX)(v) + q2_q cX(v).$$

So from (4.1), (4.2), (4.5) and (4.6):

$$\begin{aligned}
Y(\mathrm{Cas}_q(v)) &= 2_qY(b)K^{-1}Y(v) + 2_qK(b)YK^{-1}Y(v) + Y(g)(q^2XY - YX)(v)\\
&\quad + K(g)(q^2YXY - Y^2X)(v) + q2_qY(c)X(v) + q2_qK(c)YX(v)\\
&= -2_qgK^{-1}Y(v) + q^22_qbq^{-2}K^{-1}Y(Y(v)) + q^{-1}2_qc(q^2XY - YX)(v)\\
&\quad + g(q^2YXY - Y^2X)(v) + q^{-1}2_qcYX(v)\\
&= -2_qgK^{-1}Y(v) + q^{-1}2_qbX_-(Y(v)) + q2_qcX_+(Y(v)) - q^{-1}2_qcYX(v)\\
&\quad + gX_0(Y(v)) + g\left(-q^2\frac{K-K^{-1}}{q-q^{-1}}Y + Y\frac{K-K^{-1}}{q-q^{-1}}\right)(v) + q^{-1}2_qcYX(v)\\
&= \mathrm{Cas}_q(Y(v)) - 2_qgK^{-1}Y(v) + (q-q^{-1})^{-1}g(-q^2KY + q^2K^{-1}Y\\
&\quad + q^2KY - q^{-2}K^{-1}Y)(v)\\
&= \mathrm{Cas}_q(Y(v)) - 2_qgK^{-1}Y(v) + 2_qgK^{-1}Y(v) = \mathrm{Cas}_q(Y(v)).
\end{aligned}$$

□

6. Representation theory of braided tangent vector fields

Fix the parameter $\theta \in \mathbb{K}$ of (2.7). (For $q = 1, \theta = 2$.) We know how to extend B_q, G_q, C_q to $\mathbb{K}_q[\mathbb{R}^3]$ (and $\mathbb{K}_q\left[\mathrm{H}^2_\rho\right]$) preserving the relation (5.3). We want to define now which is the unique extension which preserves also the relation (2.7). For that, we apply what we know about the representation theory of $sl(2)_q$ in [DS] to X_+, X_0, X_- in V_k.

Recall that in the basis $\{b, -g, -q^{-1}c\}$ on $V_1 = \mathcal{SL}$, our vector fields B_q, G_q, C_q are defined by the matrices B_q^1, G_q^1, C_q^1:

$$B_q^1 = \tau_1\begin{pmatrix}0 & 2_q & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix} \qquad G_q^1 = \tau_1\begin{pmatrix}q^22_q & 0 & 0\\ 0 & (q^2-1)2_q & 0\\ 0 & 0 & -2_q\end{pmatrix}$$
$$C_q^1 = \tau_1\begin{pmatrix}0 & 0 & 0\\ q & 0 & 0\\ 0 & q^32_q & 0\end{pmatrix} \tag{6.1}$$

with $\tau_1 = 2_q^{-1}\omega$.

In this basis, our action of $U_q(sl(2))$ on $\mathcal{SL}$ is defined by the following matrices:

$$X = \begin{pmatrix}0 & 2_q & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}, \qquad K = \begin{pmatrix}q^2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & q^{-2}\end{pmatrix}, \qquad Y = \begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 2_q & 0\end{pmatrix},$$

and we know that

$$B_q^1 = \tau_1X = \tau_1X_+, \qquad G_q^1 = \tau_1(q^2XY - YX) = \tau_1X_0, \qquad C_q^1 = \tau_1qK^{-1}Y = \tau_1X_-.$$

This leads us to generalize this construction on V_k for $k \in \mathbb{N}^*$.

Considering the basis $\left\{b^k, \frac{1}{1_q!}Y(b^k), \frac{1}{2_q!}Y^2(b^k), \ldots, \frac{1}{(2k)_q!}Y^{2k}(b^k)\right\}$

where $i_q! = i_q \times (i-1)_q \times \cdots \times 2_q \times 1_q$, we define the operators B_q^k, G_q^k, C_q^k on V_k endowed with our basis with the $(2k+1) \times (2k+1)$ matrices:

$$B_q^k = \tau_k \cdot \mathrm{diag}_+((2k)_q, (2k-1)_q, (2k-2)_q, \ldots, 2_q, 1) \tag{6.2}$$

$$G_q^k = \tau_k \cdot \mathrm{diag}(q^2(2k)_q, q^2 2_q(2k-1)_q - (2k)_q, \ldots, q^2(2k)_q - 2_q(2k-1)_q, -(2k)_q) \tag{6.3}$$

$$C_q^k = \tau_k \cdot \mathrm{diag}_-(q^{-2k+3}, q^{-2k+5}2_q, q^{-2k+7}3_q, \ldots, q^{2k+1}(2k)_q) \tag{6.4}$$

We just used the Lyubashenko–Sudbery construction in our new basis:

$$B_q^k = \tau_k X_+ = \tau_k X, \qquad G_q^k = \tau_k X_0 = \tau_k(q^2XY - YX), \qquad C_q^k = \tau_k X_- = \tau_k q K^{-1} Y$$

with the following representation of $U_q(sl(2))$ on V_k:

$$\begin{aligned} X &= \mathrm{diag}_+((2k)_q, (2k-1)_q, (2k-2)_q, \ldots, 2_q, 1) \\ K &= \mathrm{diag}(q^{2k}, q^{2k-2}, \ldots, q^{-2k+2}, q^{-2k}) \\ Y &= \mathrm{diag}_-(1, 2_q, 3_q, \ldots, (2k-1)_q, (2k)_q). \end{aligned}$$

We deduce from the relations (5.4) and (4.7) that the operators B_q^k, G_q^k, C_q^k are subject to (5.3) and (5.2) for appropriate choices of $\tau_k \in \mathbb{K}$:

$$\forall\, k \in \mathbb{N}^*, \qquad \tau_k = \frac{q^{-2}\theta}{(2k+2)_q - (2k)_q} = \frac{q^{-2}\theta}{q^{2k+1} + q^{-2k-1}}.$$

Remark 11. We can see that $B_1^k = B^k, G_1^k = G^k, C_1^k = C^k$ if we pose $\theta = 2$.

Example 12. If $k = 2$, we consider the following basis of the vector space V_2:

$$V_2 = \mathrm{span}(b^2, -q^2bg - gb, qg^2 - q^3bc - q^{-1}cb, qgc + q^{-1}cg, q^{-2}c^2).$$

In this basis, we obtain the following representation of our braided vector fields:

$$B_q^2 = \tau_2 \begin{pmatrix} 0 & 4_q & 0 & 0 & 0 \\ 0 & 0 & 3_q & 0 & 0 \\ 0 & 0 & 0 & 2_q & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$G_q^2 = \tau_2 \begin{pmatrix} q^2 4_q & 0 & 0 & 0 & 0 \\ 0 & q^2 2_q 3_q - 4_q & 0 & 0 & 0 \\ 0 & 0 & (q^2-1)2_q 3_q & 0 & 0 \\ 0 & 0 & 0 & q^2 4_q - 2_q 3_q & 0 \\ 0 & 0 & 0 & 0 & -4_q \end{pmatrix}$$

$$C_q^2 = \tau_2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ q^{-1} & 0 & 0 & 0 & 0 \\ 0 & q2_q & 0 & 0 & 0 \\ 0 & 0 & q^3 3_q & 0 & 0 \\ 0 & 0 & 0 & q^5 4_q & 0 \end{pmatrix}.$$

The reader can now calculate what the representations $B_q^k,\ G_q^k,\ C_q^k$ on V_k for $k > 2$ are.

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